

# New Prescription in light-cone gauge theories

D. K. Park

Department of Physics, KyungNam University, Masan, 631-701 Korea

## Abstract

New prescription for the singularities of light-cone gauge theories is suggested. The new prescription provides Green's function which is identical with and different from that of Mandelstam-Leibbrandt prescription at  $d = 4$  and  $d = 2$  respectively.

The light-cone gauge(radiation gauge in light-cone coordinate), one of the latest noncovariant gauge, has been frequently used for the calculation of perturbative QCD, the quantization of the supersymmetric Yang-Mills theories, and the non-covariant formulation of string theories in spite of the lack of manifest Lorentz covariance.[1-4]

However, implementation of light-cone gauge is not entirely straightforward, at least in perturbation theory. This statement is easily described by considering the free gauge field propagator[5]

$$G_{\mu\nu}^{ab}(k) = \frac{-i\delta^{ab}}{k^2} \left[ g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{n \cdot k} \right]. \quad (1)$$

From Eq.(1) one can see that there are two kinds of singularities in  $G_{\mu\nu}^{ab}(k)$ . First singularity arises when  $k^2 = 0$ . This is the univeral property of massless fields. Usually this singularity is prescribed by choosing a causal prescription,

$$\frac{1}{k^2} \rightarrow \frac{1}{k^2 + i\epsilon}. \quad (2)$$

In  $G_{\mu\nu}^{ab}(k)$  there is another singularity which arises when  $n \cdot k = 0$ . This "spurious singularity" is peculiar one in light-cone gauge.[6]

For the last decade various prescriptions have been made for the spurious singularity. If a Cauchy principal value(CPV) prescription

$$\frac{1}{k^-} \rightarrow CPV\left(\frac{1}{k^-}\right) \equiv \frac{1}{2} \left[ \frac{1}{k^- + i\epsilon} + \frac{1}{k^- - i\epsilon} \right], \quad (3)$$

which plays an crucial role in other non-covariant gauges is choosed, calculation of the various Feynman diagrams produces a poorly defined integrals.[7]

A more successful prescription for the spurious singularity which is usually called by Mandelstam-Leibbrandt(ML) prescription

$$\frac{1}{k^-} \rightarrow ML\left(\frac{1}{k^-}\right) \equiv \frac{k^+}{k^+ k^- + i\epsilon} \quad (4)$$

is suggested independently by Mandelstam[8] and Leibbrandt[9]. Later it is proved in the framework of equal-time canonical quantization that ML prescription is nothing but the causal prescription[10] and the renormalizability of the gauge theories formulated in this way is also shown[11] although non-local counterterms are necessary to render off-shell Green's function finite.

Although attention is paid only for the spurious singularity for the last decade, there was a suggestion for  $k^2 = 0$  singularity on two dimensional light-cone about twenty years ago.[12] The authors in Ref.[12] suggested that the propagators on the two-dimensional light-cone is different from those in conventional coordinates as follows by analyzing the massless scalar and fermion theories

$$\frac{1}{k^2 + i\epsilon} \Rightarrow \frac{1}{k^2 + i\epsilon} + \frac{i\pi}{2} \frac{\delta(k^-)}{|k^+|}. \quad (5)$$

Recently it is shown [13] that the difference of the propagator on the two-dimensional light-cone also can be interpreted as the difference of prescription like

$$\frac{1}{2k^+} ML\left(\frac{1}{k^-}\right) \Rightarrow \frac{1}{2k^+} CPV\left(\frac{1}{k^-}\right). \quad (6)$$

This means that the prescription problem arises not only in the spurious singularity but also in  $k^2 = 0$  singularity in the light-cone gauge theories. In this paper, therefore, we will choose the prescriptions for  $k^2 = 0$  and spurious singularities simultaneously.

In order to find the new prescription let us consider only  $G_{--}^{ab}(k)$  which is the only non-vanishing component in two-dimensional theory. If one chooses ML-prescription

$$[G_{--}^{ab}(k)]_{ML} = \frac{2i\delta^{ab}k^+}{k^2 + i\epsilon} ML\left(\frac{1}{k^-}\right), \quad (7)$$

then  $(-, -)$  component of d-dimensional Green's function defined as

$$D_{\mu\nu}^{ab}(x) \equiv \frac{1}{(2\pi)^d} \int d^d k G_{\mu\nu}^{ab}(k) e^{ikx} \equiv \delta^{ab} D_{\mu\nu}(x) \quad (8)$$

is

$$\begin{aligned} & [D_{--}(x)]_{ML} \\ &= \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} (-x^2 + i\epsilon)^{-\frac{d}{2}} (x^+)^2 \\ &\quad \times \left[ {}_2F_1\left(1, \frac{d}{2} - 1; 2; \frac{2(n \cdot x)(n^* \cdot x)}{(n \cdot n^*)x^2}\right) + \left(\frac{\mathbf{x}_T^2}{x^2}\right) {}_2F_1\left(2, \frac{d}{2}; 3; \frac{2(n \cdot x)(n^* \cdot x)}{(n \cdot n^*)x^2}\right) \right] \end{aligned} \quad (9)$$

where  ${}_2F_1(a, b; c; z)$  is usual hypergeometric function. One can show that  $d \rightarrow 4$  limit of Eq.(9) coincides with the result of Ref.[14] if the difference of definition of  $n^\mu$  is considered. If one takes  $\mathbf{x}_T \rightarrow \mathbf{0}$  limit, Eq.(9) becomes

$$\lim_{\mathbf{x}_T \rightarrow \mathbf{0}} [D_{--}(x)]_{ML} = \frac{2\Gamma(\frac{d}{2})(x^+)^2}{\pi^{\frac{d}{2}}(4-d)} (-x^2 + i\epsilon)^{-\frac{d}{2}} \quad (10)$$

whose  $d \rightarrow 2$  limit is

$$[D_{--}(x)]_{ML}^{d=2} = \frac{(x^+)^2}{\pi} \frac{1}{-x^2 + i\epsilon}. \quad (11)$$

This is different from

$$[D_{--}(x)]'_{tHooft} = -\frac{i}{2} |x^+| \delta(x^-) \quad (12)$$

which was used by 't Hooft in Ref.[15] for the calculation of the mesonic mass spectrum. This difference makes the authors of Ref.[16] suggest that the two-dimensional pure Yang-Mills theory with light-cone gauge is not free theory. Their suggestion arises from the fact that calculational result of the vacuum expectation value of the lightlike Wilson-loop operator with ML-prescription at  $O(g^4)$  does not exhibit abelian exponentiation. So it is

worthwhile to check whether there exists a prescription which provides a Green's function whose  $d \rightarrow 2$  limit coincides with  $[D_{--}(x)]'_{tHooft}$ . Soon it will be shown that this will be achieved by choosing CPV-prescription for  $k^2 = 0$  and spurious singularities simultaneously like

$$\begin{aligned}
& G_{--}^{ab}(k) \\
&= \frac{2i\delta^{ab}k^+}{k^2} \frac{1}{k^-} \\
&\rightarrow [G_{--}^{ab}(k)]_{NCPV} \\
&\equiv i\delta^{ab}CPV\left(\frac{1}{k^-(k^--\frac{\mathbf{k}_T^2}{2k^+})}\right) \\
&= i\delta^{ab}\frac{2k^+}{\mathbf{k}_T^2} \left[ CPV\left(\frac{1}{k^--\frac{\mathbf{k}_T^2}{2k^+}}\right) - CPV\left(\frac{1}{k^-}\right) \right].
\end{aligned} \tag{13}$$

By using the formula

$$\begin{aligned}
& CPV\left(\frac{1}{k^--\frac{\mathbf{k}_T^2}{2k^+}}\right) \\
&= \frac{1}{k^--\frac{\mathbf{k}_T^2}{2k^+} + i\epsilon\epsilon(k^+)} + i\pi\epsilon(k^+)\delta(k^--\frac{\mathbf{k}_T^2}{2k^+}), \\
& CPV\left(\frac{1}{k^-}\right) \\
&= ML\left(\frac{1}{k^-}\right) + i\pi\epsilon(k^+)\delta(k^-),
\end{aligned} \tag{14}$$

$[G_{--}^{ab}(k)]_{NCPV}$  becomes

$$[G_{--}^{ab}(k)]_{NCPV} = [G_{--}^{ab}(k)]_{ML} - 2\pi\delta^{ab}\frac{|k^+|}{\mathbf{k}_T^2} \left[ \delta(k^--\frac{\mathbf{k}_T^2}{2k^+}) - \delta(k^-) \right]. \tag{15}$$

From Eq.(15)  $[D_{--}(x)]_{NCPV}$  is directly calculated and the final result is

$$[D_{--}(x)]_{NCPV} = [D_{--}(x)]_{ML} + \Delta D_{--}(x) \tag{16}$$

where  $[D_{--}(x)]_{ML}$  is given in Eq.(9) and  $\Delta D_{--}(x)$  is

$$\begin{aligned} & \Delta D_{--}(x) \\ &= -2^{2-d}\pi^{-\frac{d}{2}} \sum_{l=1}^{\infty} \frac{\Gamma(l+\frac{d}{2}-2)}{l!\Gamma(1-l)} \left(-\frac{x^+}{2}\right)^l \left(\frac{\mathbf{x}_T^2}{4}\right)^{2-\frac{d}{2}-l} (\partial_-)^{-l+1} CPV\left(\frac{1}{x^-}\right). \end{aligned} \quad (17)$$

The modification term  $\Delta D_{--}(x)$  does not give a finite contribution at  $d = 4$ . Therefore this new prescription provides a same  $D_{--}(x)$  with ML-prescription. After the calculation of other components one can show that all components of  $[D_{\mu\nu}(x)]_{NCPV}$  coincide with  $[D_{\mu\nu}(x)]_{ML}$  at  $d = 4$ .

However the situation is completely different at  $d = 2$ . In this case  $\Delta D_{--}(x)$  gives a finite contribution when  $l = 1$ . By considering this finite contribution  $[D_{--}(x)]_{NCPV}$  at  $d = 2$  becomes

$$\begin{aligned} \lim_{d \rightarrow 2} [D_{--}(x)]_{NCPV} &= \frac{(x^+)^2}{\pi} \frac{1}{-x^2 + i\epsilon} + \frac{x^+}{2\pi} CPV\left(\frac{1}{x^-}\right) \\ &= -\frac{i}{2} |x^+| \delta(x^-), \end{aligned} \quad (18)$$

which is exactly same with  $[D_{--}(x)]_{tHooft}$ . Therefore this new prescription provides a same Green's function with ML-prescription at  $d = 4$  and 't Hooft approach at  $d = 2$ . Furthermore if one follows this new prescription, one can not say that two dimensional pure Yang-Mills theory with light-cone gauge is interacting theory which is suggested in Ref.[16]. For example let us calculate the crossed diagram of lightlike Wilson-loop operator which gives a non-vanishing and vanishing contributions if one chooses a ML-prescription and 't Hooft approach respectively. After following the notation of Ref.[16] this new CPV prescription gives

$$\begin{aligned} & [W_{crossed}]_{CPV} \\ &= -\frac{1}{2}(ig)^4 \mu^{4-2d} C_F C_A (n^{*-})^4 \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_1^0 dt_1 \end{aligned} \quad (19)$$

$$\begin{aligned} & \int_1^{t_1} dt_2 [D_{--}(n + n^*(t_1 - s_1))]_{NCPV} [D_{--}(n + n^*(t_2 - s_2))]_{NCPV} \\ &= [W_{crossed}]_{ML} + \Delta W_{crossed}. \end{aligned}$$

$[W_{crossed}]_{ML}$  is already calculated in Ref.[16];

$$\begin{aligned} & [W_{crossed}]_{ML} \\ &= -\left(\frac{g}{\pi\mu}\right)^4 \frac{C_F C_A}{16} \frac{\Gamma^2(\frac{d}{2}-1)}{(d-4)^2} \left[ 2A \frac{d-2}{d-3} + 8B \left( 1 - 2 \frac{\Gamma^2(3-\frac{d}{2})}{\Gamma(5-d)} \right) \right], \end{aligned} \quad (20)$$

where

$$\begin{aligned} A &= (2\pi\mu^2 n \cdot n^* + i\epsilon)^{4-d} + (-2\pi\mu^2 n \cdot n^* + i\epsilon)^{4-d}, \\ B &= [(2\pi\mu^2 n \cdot n^* + i\epsilon)(-2\pi\mu^2 n \cdot n^* + i\epsilon)]^{2-\frac{d}{2}}, \end{aligned} \quad (21)$$

and this gives a finite contribution at  $d = 2$

$$\lim_{d \rightarrow 2} [W_{crossed}]_{ML} = \frac{g^4}{48} C_F C_A (n \cdot n^*)^2. \quad (22)$$

In order to calculate  $\Delta W_{crossed}$  we divide it as two parts

$$\Delta W_{crossed} = \Delta W_1 + \Delta W_2 \quad (23)$$

where

$$\begin{aligned} \Delta W_1 &= -\frac{1}{2} (ig)^4 \mu^{4-2d} C_F C_A (n^{*-})^4 \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^1 dt_1 \int_{t_1}^1 dt_2 \\ &\quad \left[ [D_{--}(n + n^*(t_1 - s_1))]_{ML} \Delta D_{--}(n + n^*(t_2 - s_2)) \right. \\ &\quad \left. + [D_{--}(n + n^*(t_2 - s_2))]_{ML} \Delta D_{--}(n + n^*(t_1 - s_1)) \right] \end{aligned} \quad (24)$$

and

$$\begin{aligned} \Delta W_2 &= -\frac{1}{2} (ig)^4 \mu^{4-2d} C_F C_A (n^{*-})^4 \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^1 dt_1 \int_{t_1}^1 dt_2 \\ &\quad \Delta D_{--}(n + n^*(t_1 - s_1)) \Delta D_{--}(n + n^*(t_2 - s_2)). \end{aligned} \quad (25)$$

After tedious calculation one can show that  $\Delta W_1$  and  $\Delta W_2$  provide finite contribution to  $[W_{crossed}]_{NCPV}$  at  $d = 2$

$$\lim_{d \rightarrow 2} \Delta W_1 = -\frac{g^4}{24} C_F C_A (nn^*)^2 \quad (26)$$

$$\lim_{d \rightarrow 2} \Delta W_2 = \frac{g^4}{48} C_F C_A (nn^*)^2$$

from which the vanishing of  $[W_{crossed}]_{NCPV}$  can be proved. This result is in agreement with that of 't Hooft approach and differs from that of ML-prescription. So it is worthwhile to calculate the remaining  $O(g^4)$  diagrams(self-energy and vertex diagrams) to check whether the two-dimensional Yang-Mills theory with light-cone gauge is free or not by using this new prescription. This work will be reported elsewhere.

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## **References**

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[5] My conventions are

$$\begin{aligned}x_{\pm} &= \frac{1}{\sqrt{2}}(x_0 \pm x_{d-1}) & \mathbf{x_T} &= (x_1, x_2, \dots, x_{d-2}) \\x_{\pm} &= x^{\mp} & n^2 = n_+ = 0 & \quad n^{*2} = n_-^* = 0\end{aligned}$$

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